Part II:

THE DIFFERENCES BETWEEN' MISCONCEPTIONS', 'ERRORS' AND 'MISTAKES'

In this second part of Volume 2 we focus on the meaning of the word 'misconceptions' and compare it to the general understanding of the meaning of 'errors' and 'mistakes'. These different words have different concepts associated with them. We tend to use the word 'misconception' indiscriminately when talking to colleagues, and more often than not, we are talking about learner errors or mistakes rather than misconceptions.

As we discussed in Newsletter issue 1 last year, the idea of 'misconceptions' describes the alternative understandings that people may have about concepts. An important point to remember when working with misconceptions is that they often arise through perfectly reasonable thinking by the learner and make sense to the learner. We as teachers are able to identify misconceptions in our learners through understanding *why* they have made the errors that we see. Errors are often repeated in different contexts because the learner has an underlying misconception. We shake our heads and wring our hands over learners who keep making the same errors over and over again, even though we have addressed them repeatedly in class and corrected them.

INSIDE THIS ISSUE

When incorrect conceptual understanding makes sense to the learner, he or she must realise that flaws exist in this understanding before he or she can restructure the knowledge. *Understanding where these errors are coming from and **why** they are being made is a very valuable tool that we may use to improve our teaching and improve learner understanding of concepts.* How teachers can use errors for the benefit of their teaching and improvement of learner understanding will be discussed in Newsletter 3.

Mistakes, on the other hand, are incorrect answers that have been obtained through slips, for example, a slip made in a calculation, such as $3 \times 3 = 6$. A mistake is quickly realised and corrected because there is no deep conceptual misunderstanding associated with the mistake. It is important that as teachers we are able to distinguish between mistakes (easily rectified) and errors (more difficult to address), so that we can make better teaching decisions.

Identifying the misconceptions that lie behind the errors that learners make is difficult to do without a fair amount of research. The DIPIP team is focusing on certain errors that were made by large numbers of learners in the ICAS tests. As we identify significant misconceptions that seem to stretch across many grades, we are reporting back to you.

WHAT THE LEARNERS DID

The ICAS tests for grades 7, 8 and 9 had 3 out of 40, 2 out of 40, and 4 out of 40 items respectively that were directly related to working with number sentences and equations. Except for one grade 7 item which had an achievement of 71%, between 0% and 33 % of grade7, 8 and 9 learners obtained the correct answers for these 8 questions, showing an overall poor performance in the area of number sentences and equations.

We have discussed in detail in Newsletter 1A the problems many learners have when working with number sentences. If they do not have a deep understanding of the equal sign, they tend to understand its presence as a command to 'do something'. This is one of the main reasons why learners subsequently struggle to understand how to solve equations. We argued that learners need to understand the **relational** meaning of the equal sign; where one side of the equal sign is related to the other side through equality. Once they understand this, we may then move on to explanations of solving for *x*; opposite, or inverse, operations; inequalities and proofs.

Below are the grade 8 and 9 ICAS items we chose to demonstrate issues around solving equations. To remind you, they are based on the NCS LO2:
Grade 8: *'… when the learner … **solves equations** by inspection, trial-and-*

*improvement, or algebraic processes (**additive or multiplicative inverses**), checking the answer by **substitution**.'*
Grade 9: *'… when the learner … **solves equations** by inspection, trial-and-improvement, or algebraic processes (**additive or multiplicative inverses, and factorisation**), checking the answer by **substitution**.'*

As before, we provide an ICAS question and an analysis of learner performance for that question. Next we discuss the correct answer as well as the different ways learners might have used to obtain the incorrect answers. We then move into a discussion of one or two of the misconceptions related to the ICAS items and the associated assessment standard.

Grade 8: 'Solve a simple linear equation that has one unknown variable'

$\frac{1}{2}(y+3)=8$
What is the value of y*?*

(A) 13   (B) 10   (C) 7   (D) 1

Only 20% of learners arrived at the correct answer (A)
17% chose option B
28% chose option C
33% chose option D
1% of learners did not answer this question

To do this item correctly learners may use a number of procedures. Firstly, they may multiply both sides of the equation by 2. This would be the most efficient way to simplify the equation. For this the learner would need to understand that $\frac{1}{2}(y+3)$ is the same as $\frac{(y+3)}{2}$, so that the quickest way of isolating the variable, *y*, on one side of the equation is by applying the

multiplicative inverse of $\frac{1}{2}$; which is 2. The equation would then be: $y + 3 = 16$. Applying the additive inverse of '+3' means that 3 would be subtracted from both sides of the equal sign, resulting in the answer of $y = 13$. Additionally, the learner would need to recognise that $\frac{1}{2}(y + 3)$ is a representation of the distributive law, and that the whole bracket is multiplied by $\frac{1}{2}$. Therefore, to 'remove' the $\frac{1}{2}$, the whole bracket has to be multiplied by 2, and the same thing done to the other side of the equal sign.

A second method could be to apply the distributive law. Learners may multiply the whole bracket by $\frac{1}{2}$, to get $\frac{y}{2} + \frac{3}{2} = 8$. They may multiply both sides of the equation by the LCM, which is 2; which would give $y + 3 = 16$ as above, and then solve as above. Alternatively they may do the following:

$$\frac{y}{2} + \frac{3}{2} = 8 \qquad OR \qquad \tfrac{1}{2}y + 1\tfrac{1}{2} = 8$$

$$\therefore \frac{y}{2} = 8 - \frac{3}{2} \qquad\qquad \tfrac{1}{2}y = 8 - 1\tfrac{1}{2}$$

$$\therefore \frac{y}{2} = \frac{16 - 3}{2} \qquad\qquad \therefore \tfrac{1}{2}y = 6\tfrac{1}{2}$$

$$\therefore \frac{y}{2} = \frac{13}{2} \qquad\qquad \therefore y = 13$$

$$\therefore y = 13$$

We suggest that the solution on the right shows the most understanding.

A third way to solve the question is to substitute into the equation the four choices given in the list of possible answers, A to D. Although this might seem like an easy strategy, there is the need to understand the concept of equality of both sides of the equal sign and also know how to correctly substitute values into a variable to check an answer. Thus, since the left hand side (LHS) of the equal sign is equal to the

right hand side (RHS), the value of $y = 13$ makes the equation stay true and is therefore the correct answer:

$$LHS = \tfrac{1}{2}(13 + 3)$$
$$= \tfrac{1}{2}(16)$$
$$= 8$$
$$= RHS$$

A fourth method of solving this problem would be to solve it by inspection – not requiring the 4 given choices A – D. It is feasible that they could mentally do the following:

$$\tfrac{1}{2}(y + 3) = 8$$
$$\Rightarrow (y + 3) = 16$$
$$\Rightarrow y = 13$$

Solving the equation this way implies that these learners have a strong understanding of the equal sign.

However, since only 20% of learners correctly answered this item, we can safely say that learners had a problem using any one of the above methods to solve for *y*.

Analysis of learner errors showed the following common errors. A third of learners obtained an answer of *y* = 1. The following procedure is very logical to many:

(1)

$$\tfrac{1}{2}(y + 3) = 8$$
$$\therefore \frac{\tfrac{1}{2}(y + 3)}{\tfrac{1}{2}} = \frac{8}{\tfrac{1}{2}}$$
$$\therefore y + 3 = \frac{8}{\tfrac{1}{2}}$$
$$\therefore y + 3 = 4$$
$$\therefore y = 4 - 3$$
$$\therefore y = 1$$

Lines 2 and 3 are correct; the problem arises in line 4. Where lines 2 and 3 actually mean '8 divided by $\frac{1}{2}$', learners often read

it as 'half of 8', which is 4 (see line 4). Problems with Line 5 will be discussed with the choices of the other distracters.

Almost a third of learners said that $y = 7$ is the correct answer. The following possible solution indicates a lack of understanding of opposite operations, or additive and multiplicative inverses; probably because learners are attempting to follow the 'rules' to solve for y.

$8 \div \frac{1}{2} = 4$ was discussed above.

$$\therefore \frac{\frac{1}{2}(y+3)}{\frac{1}{2}} = \frac{8}{\frac{1}{2}}$$
$$\therefore y + 3 = 4$$
$$\therefore y = 4 + 3$$
$$\therefore y = 7$$

The additive inverse of +3 is –3. $\therefore$ 3 should have been subtracted from both sides.

If learners chose $y = 10$ as the correct answer, they could have done the following:

$$\frac{1}{2}(y+3) = 8$$
$$\therefore \frac{1}{2}y = 8 - 3$$
$$\therefore \frac{1}{2}y = 5$$
$$\therefore y = 5 \times 2 = 10$$

A solution such as this one is common. It has many variations, but we will focus on the one above. There are two problems here. First, the learner does not understand the distributive law – the brackets were disregarded and the learner removed the +3 from the bracket and moved it to the other side of the equal sign, in an attempt to apply opposite operations to the equation. Ignoring the brackets and misunderstanding the distributive law are common problems among learners. The second problem is related to this, in that the learner does not

understand the order of operations. In the first line, 3 is added to y before they are multiplied by $\frac{1}{2}$. The learner does not understand that in order to "undo" this, the equation must be multiplied by 2, and then 3 subtracted from both sides. A double problem such as this one, relates to different assessment standards within the NCS LO1.

A third method may have been used to obtain an answer of $y = 10$. If substitution of one of the four choices given was used as a technique to find the correct answer, then:

$$\frac{1}{2}(y+3) = 8$$
$$\text{substitute } y = 10$$
$$LHS = \frac{1}{2}(10+3)$$
$$= 5 + 3$$
$$= 8$$
$$= RHS$$
$$\therefore y = 10$$

Again the distributive law is not applied correctly in this example. Learners who do not understand the distributive law will not realize that the number outside the bracket indicates that **both** the numbers inside the bracket should be multiplied by that number.

A significant problem experienced by learners and teachers is one of solving equations using 'rules'. Phrases used to teach the solution of equations, such as 'taking the number to the other side', 'jumping over', 'hopping over', or 'transposing' leave the learner with a very confused idea as to what is really taking place in these situations. Line 5 of solution (1) on the previous page does not show what is really occurring:

$$\therefore y + 3 = 4$$ — Subtract 3 from LHS

$$\therefore y + 3 - 3 = 4 - 3$$

$$\therefore y = 4 - 3$$

$$\therefore y = 1$$ — Also subtract 3 from RHS

The second line is usually not emphasised and is generally left out when we solve the equation. But a danger is that we do not help the learners to understand *why* they are putting a -3 on the right hand side. That is, the concept of keeping both sides of the equal sign equal, understanding of the meaning and use of 'opposite operations', as well as understanding of what the equal sign represents, is often missed by the learner; and a set of rules is learned instead.

Grade 9: Solve a linear equation with one unknown

What is the value of x in the following equation?
$$5x + 6 = 11 - 2x$$

(A) $\frac{5}{7}$ **(B)** $\frac{7}{5}$ **(C)** $\frac{17}{7}$ **(D)** $\frac{5}{3}$

33% of learners chose correct answer (A)
21% chose option B
19% chose option C
24% chose option D
3% of learners did not answer this question

This question is more difficult for the learners than the grade 8 question because it has x on both sides and cannot easily be solved by inspection.

The correct method entails gathering 'like terms' and re-writing the equation with the terms containing x on one side of the equal sign and the numbers on the other:

$5x + 2x = 11 - 6$. (There is a tendency to give learners the impression that the specified variable must be on the left hand side of the equal sign, and the other numbers and variables on the right. Of course, we know that the equation may just as correctly be written as $11 - 6 = 5x + 2x$, so we need to give the learners examples like this).

Next, we simplify the expression by adding the like terms: $7x = 5$ (or $5 = 7x$); and solve for x by using inverse operations – namely the multiplicative inverse – by dividing both sides by 7: $x = \frac{5}{7}$. (or $\frac{5}{7} = x$)

Essential conceptual understanding required for this process includes understanding the meaning of the equal sign when written between two expressions, and the related concept of using opposite operations in order to keep the equation equal, or 'true'. Linked to this is the understanding of why like terms must be gathered for successful solving of the equation.

A second technique that may be used by learners to solve this problem is that of substitution of the four choices given in the list of possible answers, A to D. The same principles as those discussed for the grade 8 item above apply here, to obtain the following for the correct value of $x = \frac{5}{7}$:

LHS	*RHS*
$= 5(\frac{5}{7}) + 6$	$= 11 - 2(\frac{5}{7})$
$= \frac{25}{7} + \frac{42}{7}$	$= \frac{77}{7} - \frac{10}{7}$
$= \frac{67}{7}$	$= \frac{67}{7}$

$$\therefore \text{LHS} = \text{RHS}$$

Solving the problem in this way shows that the learner may understand the principle of equations, but may not be confident in solving them procedurally.

Only a third of the learners answered this item correctly. For incorrect answers C, D and B the following were possible solutions:

C:
$$5x + 6 = 11 - 2x$$
$$5x + 2x = 11 + 6$$
$$7x = 17$$
$$x = \frac{17}{7}$$

D:
$$5x + 6 = 11 - 2x$$
$$5x - 2x = 11 - 6$$
$$3x = 5$$
$$x = \frac{5}{3}$$

B:
$$5x + 6 = 11 - 2x$$
$$5x + 2x = 11 - 6$$
$$7x = 5$$
$$x = \frac{7}{5}$$

All of these solutions have a common problem: that of incorrect use of opposite operations. In solution C the concept of the additive inverse of +6 was not applied in order to bring it to the right hand side and maintain equality of both sides of the equal sign although the additive inverse of -2x was used correctly. In solution D the additive inverse was incorrectly applied to the first line of the equation, this time to the -2x instead of to the +6. In solution B the multiplicative inverse was used incorrectly, so that both sides were divided by 5 instead of 7, and the answer written as $x = \frac{7}{5}$

Answering either C or D shows a lack of consistency on the learner's part. Both solutions show one additive inverse done correctly and the other done incorrectly. Why would the learner choose one in preference over the other? This could be a slip or it could be a more serious error, and is likely to be seen when the learners do not understand the concept behind using opposite operations to solve equations. Often, when solutions to equations have been learned as a set of rules, the sign is not changed when the value '*hopped*' or '*jumped*' or '*was transposed*' over the equal sign, to use some of the language commonly used by many teachers. Likewise, use of the multiplicative inverse becomes problematic for the learner attempting to get a solution by following a set of rules. What to do with the 7 and 5 at the end of the problem becomes just a confusing decision to make, rather than a decision based on the understanding of how correct use of the multiplicative inverse allows the unknown to be isolated, and the equality of the equation to be maintained.

We have identified similar errors in other ICAS items from grades 8 and 9: incorrect or incomplete understanding of the meaning of the equal sign; and incorrect, incomplete or total lack of understanding of opposite operations; most likely through the teaching of these concepts as a set of rules for solving equations. We suggest that if learners are encouraged to understand the equality of equations and to test and reject or verify their solutions to problems, they will hopefully develop a habit of reflecting on their thinking and begin to spot their own errors – a desirable component of the learning process.

HANDLING THE IDENTIFIED ERRORS AND MISCONCEPTIONS

We reported a few different misconceptions and errors in our analysis of ICAS items in the last section. Incorrect or

incomplete _understanding of the equal sign_ was addressed in detail in Newsletter 2A. We suggest that grade 7 – 9 teachers do the activity sheets in Newsletter 2A with your learners, which will help you to identify the misconceptions from previous grades. We will now discuss the _incorrect use of opposite operations_ or additive and multiplicative inverses. We will make suggestions about how to teach so that learners can overcome the identified misconceptions and improve their understanding in this area. Note that all these issues discussed are usually interwoven with each other, and we cannot always isolate one error for teaching purposes.

Opposite operations and laws for simplifying

As a follow-on from realising that our learners generally need a more 'holistic' understanding of the equal sign, it is perhaps easier to understand why they continue to use opposite operations incorrectly and inconsistently. We saw from Essien and Setati that an understanding of the relational meaning of the equal sign will help learners with the transition from completing number sentences to solving equations.

Two interrelated issues are important for learners. One is that they need to have a relational understanding of the equal sign. Another is that they need to have a deep understanding of BODMAS and the reverse of BODMAS; otherwise known as 'opposite' or 'inverse' operations. If the operations used for solving equations were taught as a set of rules, then a basic linear equation may be solved as many times incorrectly as correctly. For example, solving $\frac{1}{2}(y+3)=8$ requires 'opposite operations' to be applied.

But how does the learner know where to start, in order to solve for y? If opposite operations are not understood, it will be as logical to subtract 3 from both sides as it will be to multiply both sides by 2 as the first step to solving for y. Let us look at using opposite operations in order to solve for the unknown and to keep the equation 'balanced'.

In the following expression certain operations have been done to a variable: $3x-2$. We understand by reading this that the unknown number was first multiplied by 3 and two was subtracted from the answer. (There is an implicit understanding that BODMAS has been used to interpret the expression from symbols to words). If we then **undo the operations** that were done to the variable, we will be able to find out what the value of the variable was before it was operated on. We would have to **undo the operations**, starting at the last operation that was done, and then working backwards. That is, add 2 first, and then divide the answer by 3. We will then obtain the original value of x. If this expression is now given a specific value, say 19, in $3x-2=19$, then we can state that **if the variable is operated on, we obtain a result** of 19. That is, 'if the variable is first multiplied by 3 and then 2 is subtracted from the answer, then the result is 19'. If the learner understands fully that the equal sign in an equation means that two expressions are equal, then they should understand that numbers and variables may be 'moved' to the other side of the equal sign, but that **equality must be maintained: the equation must stay 'balanced'**. A way to keep the equation **balanced** is to "**move numbers and variables around**" by **undoing the operations** that have been done to the variable.

In order to "move numbers and variables around" we use 'opposite operations'. The

opposite of adding 4 is subtracting 4. If you add 4 to x you get $x + 4$. If you then subtract 4 from $x + 4$, you get $x + 4 - 4$, which is x, and you are back to where you started. The additive inverse of 4 is –4, and the additive inverse of –4 is 4. Likewise, the multiplicative inverse of 2 is $\frac{1}{2}$ and the multiplicative inverse of $\frac{1}{2}$ is 2. For example, if we halve a number we will get $\frac{1}{2}x$. To get back to the whole x from half of x we will need to double it to 'undo' the halving: $(\frac{1}{2}x) \times 2$ gets us back to our original x. Or, if we multiply x by 2 to give us $2x$, we will need to halve it again to undo the doubling, to get back to our original x: $(\frac{1}{2}$ of $2x = x)$.

A more complex expression we can work with is $\frac{1}{2}x + 2y - 3$. If we wanted to know how to undo the operations done to x, we would first add 3, then subtract $2y$ from the expression, and lastly multiply x by 2. If we wanted to know how to isolate y, we would have to first add 3, then subtract $\frac{1}{2}x$, then divide y by 2, in that specific order. In this example, the operations we do to the expression depend on which variable we want to isolate. Now, we know that you know this, but do your learners know it? Can they explain these concepts to you? It is even more difficult to **relate opposite operations, relational equal signs, and testing by substitution to each other** to result in a deep understanding of solving equations.

Because of these difficulties we suggest you do some specific activities with your learners to focus on a deeper understanding of how opposite operations are linked with the relational meaning of the equal sign. Encourage your learners to explain more about what they are doing while they are working with solving equations, and listen to, and work with their explanations. In this way, using your feedback, they can reflect on the choices they make when using opposite operations and gain deeper understanding. We have provided an activity sheet at the end of this Newsletter to give you some ideas of how to do this. You can make up other similar questions for yourselves. The objective is to encourage verbal and/or written explanations from the learners, and not only to give them lots of practice in solving equations. They will not necessarily gain more meaning from doing more examples.

We have tried to emphasise the importance of deep understanding of solving equations and the related issues. If learners understand these concepts, they should be able to move easily into more complex related areas, such as solving quadratic and simultaneous equations and inequalities, doing proofs, and constructing formulae to solve problems.

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SOME LEARNER ACTIVITIES

Using the Activity Sheet

The following set of tasks has been designed to encourage the learner to think systematically about how variables may be operated on, and how inverse operations are used to isolate the variable that was operated on. We have asked learners to explain their understanding in words, so that you can determine if they understand what they are doing or not.

Nico, a grade 7 group leader, in action

Tracey, Natalie, Elna and Joeby (grade 7) making some teaching aids

The grade 7 and 8 teachers mapping items and preparing teaching materials

Shadrick, Lisa, Steve, Bennita, Mandla, Tshepo and Setswakai discussing issues around the equal sign in the grade 9 classroom

Edward, Louise, Ari and Kerry (grade 8)

Activity Sheet 5

Solving Equations

1. For each of the number sentences below do the following:

i) Describe, **in the correct order**, **in words**, the operations that were done to the variable

ii) Describe, **in the correct order, in words**, how to undo these operations, so that you can get back to the variable on its own.

    a) $\frac{1}{2}(x+3)-2$

    b) $\dfrac{a+5}{3}-8$

    c) $2(4x-5)+7$

    d) $\dfrac{2p-6}{4}+13$

    e) $-3(5y+4)-15$

iii) Determine algebraically the value of the variable, by correctly using opposite operations.

    a) $\frac{1}{2}(x+3)-2=10$

    b) $\dfrac{a+5}{3}-8=50$

    c) $2(4x-5)+7=18$

    d) $\dfrac{2p-6}{4}+13=12$

    e) $-3(5y+4)-15=-90$

2. For the following expression explain in words the operations that were done to x, in the correct order, and then explain what you would have to do to isolate x. Then do the same for y.

$$\frac{x}{4}-3y+13$$

3. For which value of x is the following statement true?

$$15-\frac{2}{3}x=4(x+5)-1$$

4.   What value of x keeps the following equation true?
$$6x + 5 = -2(2x - 5) + 7$$

5.   Given: $\boxed{? \; ? \; ?}$ = 48 ÷ 4. State **at least four** different expressions that can be written in the place of $\boxed{? \; ? \; ?}$

6.   Given: $x - y = 9 \times 4$. Give at least **two** values each for x and y that will keep the equation true.

7.   If $y = 4 - 3x$, and x is any whole number between 0 and 10 inclusive, what possible values may y have? Explain in words why y can have all these values.

Teachers, you can set similar questions to give to your learners, if you have found any if the above questions particularly useful.

Try the PUZZLE on the next page…

PUZZLE

Try this Logic Problem …

MURDER, HE WROTE
Logic Problem

⭐⭐

(by Mrs. Teri Nutton)

Five authors have just sent their latest murder stories to the publishers - so we all look forward to reading them soon. In the meantime, however, we intend to completely spoil your enjoyment of the novels, by inviting you to solve the problem of who murdered whom, as well as the motive involved and the location of the story!

1. Neither the butler nor the plumber committed the murder (which took place in Brighton) for the sake of an inheritance.
2. The revenge killing didn't take place in Fishguard or Dunoon. The artist didn't murder the partner (who was neither the victim killed in revenge nor the one murdered as the result of a power struggle).
3. The dentist murdered a cousin (but not for revenge or love) in Halifax.
4. The sister wasn't murdered in Brighton or Fishguard; and the victim in Fishguard wasn't the one killed for the love of someone. The butler didn't murder his partner.
5. In the novel in which the solicitor murders someone, the motive is power, but didn't involve the killing of a friend.

SOLUTION TO PUZZLE 1

The solution is A=3; M=9; L=2; B=5; C=1.

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C=1     A=3    L=2    M=9
        B=5    A=3    A=3
        B=5    A=3    A=3
_____
   2     3      9      5
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- C can only be 1 or 2; *since the thousand digit in the sum is only 2*
- Assume its 2, then we cannot take any 'carry-overs' from hundred digits. \ B can only be 1, but that would mean that A=1 *to get* **3; Since A cannot be equal to B (given); then A cannot be 1, given B=1. Therefore C cannot be 2 and so, C can only be 1.**
- Now A cannot be 1, given that the only possible value for C is 1.
- Assume that A=2, then M can only be 1. Given that C is 1, M cannot be 1.
- Assume that A=3, then M can only be 9 to get 15; L=2 to get 9. Given that A=3, the only possible value for B is 5 to get 13 and 'carry' 1.

Do you have suggestions or ideas or teaching experiences you would like to share with us? If so, we would like to hear from you! Write to Lynne (lynette.manson@wits.ac.za) and Ingrid (Ingrid.sapire@wits.ac.za) .