Wits School of Education & Gauteng Department of Education

Maths 4 Teachers

Volume 2 Issue 1A January 2009

WELCOME BACK…

In this edition of the DIPIP Newsletter we have chosen to discuss a section of algebra that confuses many of our learners – that of solving number sentences and equations, which falls into the National Curriculum Statement Learning Outcome 2 (LO2): *Patterns, functions and algebra*. Teachers sometimes think it is easier for learners to learn how to solve equations if we teach them the 'steps for solving'. We are tempted to give our learners lots of examples to practise, so that they can master this difficult section. Just as we think they might have 'got it', they write a test and write incorrect steps in their solutions, that they 'got right' when they did the practice examples. We hope to give some insight into why this might be happening and some suggestions on how to deal with this situation. We will approach our discussion from the perspective of learners' **understanding** the solution of equations, rather than following rules. We also provide an activity sheet that you may use or adapt to help learners to better understand this section of mathematics.

In our first Newsletter we discussed the way in which learner errors can be very informative about the misconceptions the learners might have. In this issue we will analyse and discuss misconceptions relating to equations. We hope that this will deepen your understanding of the important role misconceptions can have in helping learners to improve their knowledge and understanding of mathematical concepts.

INSIDE THIS ISSUE

Marlien, Shirley and Bronwen busy with error analysis.

Jackie and Rasheed working on aligning the ICAS tests with the NCS

MORE ABOUT MISCONCEPTIONS AND THEIR USEFULNESS

Partial or erroneous thinking among learners is normal and we should expect misconceptions to surface in the learning process. We need to recognise that they are productive, and are **valuable** for showing us that learners are engaged with, and reasoning through, the content. Misconceptions are formed in any classroom under the best teaching practices. However, as we said before, they are **difficult to correct**, and if we expect to correct learners' misconceptions without involving them in the process, it is likely that we will fail.

So how can teachers and learners be involved in the process? Teachers need to understand the misconceptions that may arise in their classrooms and think of ways of addressing them in the long term. We also commented in the previous issue that '*as teachers we need to keep a close watch on how we speak about and introduce mathematical ideas*.' **Listening** closely to what learners say also helps teachers to firstly identify existing misconceptions, and secondly direct questions and activities towards correcting misconceptions.

This newsletter should help in illuminating some of the common misconceptions that exist with respect to number sentences and equations, as well as provide some suggestions about addressing these misconceptions while teaching. We will discuss how teaching using **steps, procedures and rules** can be problematic. It is an approach considered as preferable by many teachers, because it teaches learners *what to do*; but it does not help in dealing with misconceptions and helping learners to understand equations.

THIS ISSUE

This edition has been split into two parts. Issue 2A focuses on the problems that learners may experience with working with number sentences; and Issue 2B continues with a discussion about solving linear equations. You should read both newsletters together. We have structured it this way because there are very important concepts in both areas, and we want to address them thoroughly. We believe that if learners understand number sentences fully, then their subsequent learning of equations should go more smoothly.

We aim to enable you to understand how to address problem areas with your learners. Issue 2A mostly addresses the primary school, but it is crucial for all senior phase teachers to understand where your learners may have developed misconceptions in earlier grades. Likewise, Issue 2B mostly addresses senior phase teachers, but needs to be read and understood by intermediate phase teachers, so that you understand how your teaching can help learners in later years.

WHAT THE LEARNERS DID

In this section we analyse learners' performance in number sentence items in the ICAS tests. Learner performance on number sentence items ranged from 19%–67% correct in grades 4, 5 and 6, showing an overall poor performance in this particular assessment standard.

Number sentences and equations are related and are addressed in the NCS in LO2. The specific assessment standards (AS's) outline the concepts as follows:

Grades 4, 5, 6 & 7:
*'... when the learner ... **solves or completes number sentences** by inspection or trial-and-improvement, checking the answer by substitution.'*

Grade 8:
*'... when the learner ... **solves equations** by inspection, trial-and-improvement, or algebraic processes (additive or multiplicative inverses), checking the answer by substitution.'*

Grade 9:
*'... when the learner ... **solves equations** by inspection, trial-and-improvement, or algebraic processes (additive or multiplicative inverses, and factorisation), checking the answer by substitution.'*

Learner competence in number sentences, in grades 4 – 7, is an important step in the process of learning about the solution of linear equations, learned in grades 7 – 9. Newsletter 2B will deal with the solution of linear equations and their associated errors and misconceptions.

In this Newsletter we include items from grade 6 and 7 and a brief analysis of student performance on each, so that you can see more clearly the points we are making about number sentences. We discuss some of the issues around the errors made in each item:

Jenny and Berdinah (grade 3) preparing teaching material

<u>Grade 6: 'Supply the missing numbers to solve an order of operations problem'</u>

$$\triangledown ? - \boxed{?} \div 2 = 5$$

Which of these makes the number sentence true?

(A)	$\triangledown ?$ ***= 6***	$\boxed{?}$ ***= 2***
(B)	$\triangledown ?$ = 14	$\boxed{?}$ = 4
(C)	$\triangledown ?$ = 18	$\boxed{?}$ = 8
(D)	$\triangledown ?$ = 20	$\boxed{?}$ = 10

19% of learners arrived at the correct answer (A)
19% chose option B
15% chose option C
42% chose option D
3% of learners did not answer the question

For incorrect choices B, C and D – if BODMAS is ignored – then all three distractors obtain the answer of 5:

B: 14 – 4 ÷ 2 = 5

C: 18 – 8 ÷ 2 = 5

D: 20 – 10 ÷ 2 = 5

We could not determine why nearly half of the learners chose incorrect option D, but working with factors of 10 may have been more attractive to these learners.

To obtain the correct answer requires a correct understanding of number sentences, the equal sign and BODMAS. We will first discuss the role of BODMAS and then the equal sign.

BODMAS is an important rule to know when doing this kind of calculation. While a question such as this one is perfectly legitimate in this test, the DIPIP teachers pointed out that when teaching learners to do this kind of question we should probably use brackets around the division operation of the number sentence at first, to encourage learners to focus on the division part before the subtraction. Once the concept of BODMAS has been understood using brackets first, and thereafter no brackets, only then should placeholders be introduced, so that the learners can focus on the unknowns without still being side-tracked by the ordering of operations.

How does the equal sign feature in questions such as this one? The comments here are brief, because general issues with the equal sign are discussed in detail in the next section. Teachers often do not realise that they are not giving their learners a broader understanding of what the equal sign can represent. Many problems given to learners, such as $537 \times 4 = \boxed{?}$ or $900 - 358 = \boxed{?}$

impress on both teacher and learners alike that they must 'do something' and an answer is then obtained for the calculation. Have you exposed your learners to the *relational meaning* (see next section) of the equal sign?

For this grade 6 item containing two placeholders, the meaning of the equal sign has shifted from a 'do something' sign to a 'relational' sign, signifying that the relationship between the left and right hand sides of the sign has significant meaning – they are *equal*. The issue that we want to point out here, and then leave the rest of the discussion to the next section, is that if the learners have not been exposed to the relational meaning of the equal sign before now, then they will possibly have difficulty interpreting the question.

Sakhumzi, Nico, Royston and Euphodia (grade 7) discussing errors

<u>Grade 7: 'Use subtraction to find a missing number'</u>

What is the missing number?

123 + ? = 131

(A) **8** *(B)* 12 *(C)* 18 *(D)* 254

71% of learners arrived at the correct answer (A)
6% chose option B
9% chose option C
10% chose option D
2% of learners did not answer the question

To do this item correctly learners have to have a clear understanding that the question requires them to find out how much more 131 is than 123, or how much less 123 is than 131. Either way, an appropriate mathematical way to answer this is by subtracting 123 from 131, to obtain an answer of 8. This method implements the concept of 'inverse operations' required for the solution of equations in grade 8 onwards, and is therefore important for learners to understand. Another appropriate way of thinking is to add 10 to 123 to get 133, and count back 2 to get 131; making a total of 8 added to 123 to get 131. Alternatively, the concept of subtraction (the operational inverse of addition) may still be understood, but the calculation done by counting forwards from 123 until they reach 131, or backwards from 131 until 123. A mental record is kept of how many units were counted from 123 to 131, or vice versa, and an answer of 8 is reached. This is less elegant than the previous two solutions, but acceptable, as the concept of 'how much more (or less)' is understood and used.

To obtain incorrect answer D, the learners added 123 and 131 to get 254, probably because of the '+' sign in the question. The fact that 10% of learners chose this option is a concern because they do not understand number sentences and are blindly following operations. An important conceptual problem that many learners have with number sentences is that they do not understand all the possible meanings and uses of the equal sign. We will discuss this issue in the next section.

The incorrect answer B could be obtained if the learner mixes up what is being subtracted **from** what, especially if the learner thinks that 3 'cannot be subtracted from 1'. In this distractor 1 is subtracted from 3 in the units column, and then 2 is subtracted from 3 in the tens column; leading to an answer of 12.

$$\begin{array}{r} 131 \\ -123 \\ \hline 12 \end{array}$$

Units column: 3 – 1 = 2
Tens column: 3 – 2 = 1
∴ answer = 12

In this case the distracter has been deliberately chosen to determine how many learners are showing the common misconception that the smaller number must always be subtracted from the larger, even though the two values are mixed up in order to do so. Even so, learners have recognised that subtraction is required to get to the solution, and have shown understanding that the equal sign does not only mean 'do something'.

Incorrect answer C may possibly have been a subtraction 'mistake', but we will not discuss it in detail because we want to focus on the equal sign.

Our discussion in the next section will cover some general issues around teaching number sentences. We explain some of the misconceptions, and because so many learners struggle with these concepts, we provide some activity sheets that will help you to deal with these issues.

Two significant misconceptions identified by the DIPIP teachers when analysing the grade 7 item above are 'incorrect or incomplete *understanding of the equal sign'* and 'limited understanding of *operational inverses*'. We are going to talk about how to teach so that learners can overcome these identified misconceptions and improve their understanding in this area. Note that these misconceptions usually affect each other.

UNDERSTANDING THE EQUAL SIGN

Over the last twenty years a lot of research has been done on how school children understand the equal sign. Research has shown that a significant reason why many learners cannot understand introductory algebra or solve basic equations is because they lack a mathematically mature understanding of the equal sign. If you want to read further about the history of the equal sign and a more detailed discussion about the research that that been done on this topic, a short but comprehensive article

has been written by Essien and Setati called "Revisiting the equal sign", and can be located at the URL:
http://search.sabinet.co.za/images/ejour/saarmste/saarmste_v10_n1_a6.pdf

As Essien and Setati explain, the equal sign is commonly understood by children from as early as grade 1 as an indication to 'do something'. We will call it the **operative** meaning of the equal sign. For example, giving a child the question 4 + 9 = ☐ indicates to the child that the '=' sign means that the addition must be followed with an answer. So, ☐ = 3 + 5 would not make sense as a mathematical sentence to a 6 or 7 year old child, because the child would probably argue that the '=' is in the wrong place, because it does not follow an operation.

We want to point out at this stage that there is no problem with having an operative understanding of the equal sign – it is necessary and an important part of the learner's development. But if the learner does not come to understand that the equal sign incorporates important relational concepts, he or she will have problems in understanding aspects of algebra that are initiated in grade 7 and furthered in later grades. Even foundation phase learners can be exposed to the relational meaning of the equal sign. Instead of only using 2 + 3 = ☐, for example, we can add to a worksheet questions such as ☐ = 2 + 3 or 2 + ☐ = 5.

Essien and Setati described research done with grades 1 – 6 children where the children were given the mathematical sentence 8 + 4 = ☐ + 5. When asked to find the missing number, the learners wanted to put 12 in the box after the equal sign, because the equal sign indicated to them that they should add 8 and 4. The 5 did not usually feature in their solutions.

Teachers of grades 4 and 5 children can help their learners develop a more mature understanding of the equal sign by setting questions that require more from them. For example, as well as setting questions such as 23 + 57 = ☐, they could also set questions such as 57 – ☐ = 23 and 57 – 23= ☐ + 12. This last problem seems to be the most difficult to understand conceptually.

Look at the grade 6 ICAS item above with the two placeholders, and at how poorly learners performed on this item. Nearly half of the learners chose the incorrect answer. Being a number sentence, part of the difficulty it presents to learners is that the meaning of the equal sign has changed. Instead of requiring the learner to 'do something', it requires that numbers are inserted into the placeholders and tested with respect to whether or not they 'make' 5. The grade 7 ICAS item also requires a relational understanding of the equal sign. The learners would not have a problem with 123 + 8 = ☐, because one side of the equal sign tells them what to do to put into the other side. But the problem, 123 + ☐ = 131 requires learners to **compare** both sides of the equal sign, see how each sides relates to the other and insert a number that makes the statement true.

Understanding how to correctly answer the grades 6 and 7 items requires a 'shift' in the use of the equal sign to a **relational symbol.**

The equal sign as a **relational symbol** has attached to it the meaning that the **expressions written on either side of the equal sign are the same.** This relational meaning can be seen in a few different contexts, showing a relationship between expressions. For example,

$(a+b)^2 = (a+b)(a+b)$;

$16 \div 4 = 28 \div 7 = \frac{1}{2} \times 8$; or

$\tan\theta = \sin\theta / \cos\theta$

Teachers often assume that this shift in understanding has been mastered by learners, but Essien and Setati's research shows that this is not the case: Grades 8 and 9 learners still struggle to make sense of and solve number sentences and equations. Our analysis of the ICAS items confirms this. Research shows that for many primary school learners the equal sign is 'an automatic invitation to write the answer', but this also the case for many grade 8 and 9 learners. If our secondary school learners are still thinking this way and have not accommodated the relational equal sign into their understanding, they will struggle with understanding how to solve equations, and possibly also with why they even need to solve them in the first place! Primary school teachers can help to address this issue.

To address this problem, we need to go back to the relational meaning of the equal sign and teach it from a concrete perspective. This will help learners to progress from understanding how the equal sign operates as a 'do something' sign to include the relational meaning. Many learners are already doing this from an early age, but, perhaps we need to formalise it in the later primary school years by:

- moving from a 'box' indicating the unknown to a blank line, to a variable

- expecting learners to verbally explain their way through such problems and solutions,

This might help learners to think about not just doing calculations, but understanding *what* they are doing and *why* they are doing it. Have a look at the activity sheet. Given are some graded problems aimed at addressing incomplete understanding and bringing the learner to a more complete understanding of number sentences and equations.

In conclusion, Freudenthal in Essien and Setati states: "Learners must be taught the equal sign as holistically as possible. That is, not only should teachers focus on task-oriented use of the equal sign, (asymmetry, as in '4 + 3 =') inviting learners to 'do something' to the right side of the equal sign, but also learners should be taught the meaning of 4 + 3 = 7 in a "sense of 4 + 3 being another expression for 7. The best way to do this is through missing-addend problems like 4 + □ = 7".

Examples of other missing addend questions you could give your learners include
15 + □ = 28
□ - 32 = 45
□ + ◊ = 17 and
13 - □ = 45 ÷ 5

We hope this has been helpful to you for understanding why learners struggle as they do with working with number sentences; and also to give you an idea of some useful tasks to assess where the misconceptions and lack of understanding might be, and try to address problem areas. It does not answer all the questions you might have. If you want to raise some points for discussion please contact us.

Also read Newsletter 2B to continue the discussion.

SOME LEARNER ACTIVITIES

It is important that teachers help even young mathematics learners at grades 2, 3 and 4 levels to begin to be aware of the relational meaning of the equal sign. At this stage it is a conceptual awareness we are looking at: it is not necessary for learners to be able to explain the difference between the operational and relational equal sign. We provide activity sheets at the end of this newsletter that will help you to develop this understanding in young learners.

The activity sheets have been set up for learners who appear to struggle with understanding and manipulating number sentences. Although the tasks vary slightly in difficulty, the questions deal with basic conceptual understanding of how quantities relate to each other. They focus on different ways that the equal sign may be understood as a relational symbol. Note that we have asked learners to explain their calculations in writing. This is important because learners often believe they understand something, until they have to explain it to somebody else. This is when they realise that they have not understood properly, and now need further intervention.

Edward, Kerry and Alex mapping grade 8 test items

Steven, Ingrid and Mmatladi analysing a grade 5 item

Heather, Glynnis, Angie and Zanele discussing grade 4 concepts

Some discussion taking place in the grade 6 group

Activity Sheet 1

Introducing the relational meaning of the equal sign to grade 3 and 4 learners

Grades 3 and 4

Below is only a sample of questions. You can set others that are similar and think up different examples which will be helpful to your learners for understanding the relational meaning of the equal sign. Note that question 3 may have more than 1 correct way of writing the number sentence.

1. Fill in the box with the correct number:

a) 5 +   = 21 b)   = 18 – 3

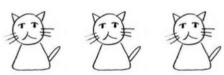
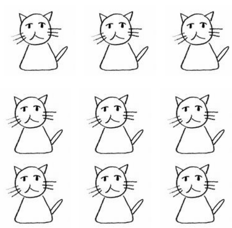
c)   - 23 = 11 d) 16 + 14 =

e) 28 -   = 12 f)   = 35 - 24

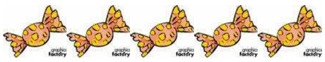
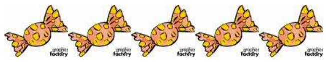
2. For each question below, first draw a picture to show the missing number of balls or cats or sweets or cup cakes, then write a number sentence underneath that describes the picture:

a)  + ________ = 

number sentence:____

b) ________________________ –  = 

number sentence:____

b) ________________ =  – 

number sentence:____

c)  + 

= ____

number sentence:____

3. For each story write a number sentence to describe the story and then work it out:

a) John had R10 but he needed to pay R15 for a toy he liked. How much money does he still need?

____

b) Nozipho borrowed R25 from her mom to go and watch a movie. She washed the dishes and cleaned the kitchen for her granny, who gave her R15. She paid her mom back with the R15. How much money does she still owe her mom?

____

c) Kara had 45 sweets and shared them evenly amongst her friends and herself. Each person received 5 sweets. How many people shared the sweets?

____

d) Mpho bought some silkworms from his friend. After the moths laid eggs and the eggs hatched he had double the number than he had previously. He now has 68 silkworms. How many silkworms did he have originally?

____

Activity Sheet 2

OPPOSITE OPERATIONS: An Introduction

Adapted from David Alcorn: Causeway Maths 1. (1995) Causeway Press, Lancashire

Grades 4 and 5

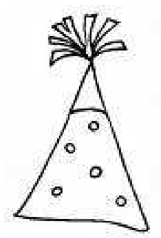
Question 5 is more appropriate for grades 5 and 6

1. Think of the opposite things that you do sometimes to 'undo' something. For example, if you spill your juice on the table, what will you do to 'undo' that?

Write down the opposites of the things that Mpho and Sipho did, shown below:

1. Mpho turned left
2. Sipho jumped up
3. Sipho ran to the gate
4. Mpho walked out of the door

2. Belinda received an invitation to a party. On the back of the invitation were the directions to the party.



Go North on the N3 highway. At the traffic lights turn left. Take the second turn right and go up the hill. At the top of the hill turn right. My house is on the left.

Belinda needs to know how to find her way home again after the party. Can you help her by giving her directions that are the opposite of what she was given? Remember to list the directions in their correct order, otherwise she won't get home!

3. Mrs Dube needs do some errands in Johannesburg. She leaves the station in Rissik Street and turns right into Smit Street, where she buys a loaf of bread at the café. She then walks down Smit Street until she reaches Klein Street, where she turns left, and immediately right into Pietersen Street.

Here she stops at the tailor to fetch and pay for a dress she had altered. She then walked to Claim Street, where she turned right, and left again into Leyds Street. There, she bought two new tracksuits for her children because the shop was having a sale. She was very hot and went around the corner into Banket Street to buy a coke.

Now she was ready to go back to the station to go home. She was going to go back using a shorter route, but realized that her train ticket must have fallen out of her bag in one of the shops she had visited. She decided to go back to the station the way she had come to see if she could find her ticket. Write down the directions she would have to go to trace her steps.

4. Belinda realized that she did not have enough money to buy the shoes she wanted. If she doubled what she had and added R7,80, she would have the right amount to buy the shoes, which cost R90. How much money did Belinda have?

5. Look at the following number sentences. List in the correct order the operations that were done to the P, and then list what you need to do to undo those operations

1. $2 \times P - 4$
2. $\frac{1}{2} \times P + 15$
3. $4\frac{1}{2} - 3 \times P$
4. $25 + 5 \times P$

Activity Sheet 3

Developing Proof: An Introduction

Grades 4, 5 and 6

Below is only a sample of questions. You can set others that are similar and think up different examples which will be helpful to your learners for understanding the concept of proof.

1. David's aunt gave David **R15** for a school charity collection. Then his father gave him a donation of **R20**. For the same charity collection Themba's grandmother gave Themba **R5**, his mother gave him **R25** and his friend gave him **R5**.

   Write down a number sentence showing how much money David was given by completing the sentence:

   David's amount = …

   Write down a similar number sentence showing how much money Themba was given.

   Write down a number sentence proving that David's and Themba's total amounts were the same.

   To the teacher: The '=' sign must be in the middle of the number sentence, with David's amount on one side and Themba's amount on the other. You have just **proved** they are the same by writing a mathematical sentence.

2. a) ☒ = (4 × 3) – 1

   ▣ = (2 × 4) + 3

   Is ☒ = ▣ ? Prove how can you be sure of your answer by showing all of your working out.

   d) ★ = 5 + (6 ÷ 2)

   ▲ = 17 – 8

   Is ★ = ▲ ? Prove your answer by showing all of your working out.

   c) ◈ = 54 ÷ 9

   ⧈ = ((7 × 3) + 3) ÷ 4

   Prove by using a number sentence that ◈ = ⧈

Activity Sheet 4
Number Sentences
Grades 4, 5, 6

1. Nozipho had R5,50 to buy some sweets. She wanted some toffees, but they cost R7,20. By how much money was she short?
   a) First write a number sentence to describe the situation
   b) Find out the amount of money that Nozipho was short, if she wanted to buy the toffees.
2. A municipality decided that they would put trees on the side of the main road in a particular suburb. They had 57 trees, but found that they were short of 26 trees. How many trees were actually needed?
   a) First write a number sentence to describe the situation
   b) Find out how many trees were actually needed for planting.
3. (i) Make up a real story situation for each of the following number sentences
   ` (ii) Then solve each one.
   (iii) Explain in words what you did to solve each problem.
   a) 563 + □ = 612
   b) □ – 14 = 197
   c) 12 = 846 – □
   d) □ + 28 = 35
   e) 1472 + 6829 = □
4. (i) Find the value represented by each □ in each of the following questions.
   (ii) Now write each of the following number sentences in <u>two</u> other different ways that keep the amounts the same and the number sentences true.
   a) 137 – □ = 98
   b) □ – 74 = 382
   c) 934 + 467 = □
   d) 582 – 245 = □
   e) □ + 589 = 734
   f) 683 + □ = 712

5. Find <u>one</u> possible value for each of □ and ∅ in each case below. (remember BODMAS!)
   a) 100 – 79 = 3 × □
   b) 25 = □ – ∅
   c) 48 – 2 × ∅ = 30
   d) 4 × □ + ∅ = 14
   e) 19 = 15 ÷ □ + ∅
   f) 2 × ∅ + 24 ÷ □ = 33
6. It is given that $3x - 4 = 32$. Can x have a value of 10? Show how you got to your answer.
7. If you know that $26 = 14 + 2p$, can p be 12? Show how you got to your answer.
8. Given: $48 \div 2a = 12$. Can a have a value of 24? Show how you got to your answer.
9. Is the following number sentence true? Explain why it is or is not in words and/or mathematical symbols and working:

$$\frac{28}{7} - 3 + 10 = \left(\frac{100}{2} + 5\right) \div 5$$

10. Is the following number sentence true? Explain why it is or is not:

$$(9 \times 5 - 3) \div 14 = (3 \times 5 \times 4) - \left(\frac{40}{2}\right)$$

11. What value of x makes the following number sentence true:

$$15 = 18 - x$$

12. What value of x makes the following number sentence true:

$$x - 72 = 51$$