Mkwawa University College

Calculus of Several Variables
(Analysis 2)

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1 Course Content Elements

1.1 Course Title / Course Code

Course Title: Calculus of several variables

Course Code: MT200

1.2 Instructor(s) Introduction

The course will be delivered by Irunde, Jacob Ismail who is assistant lecturers at mathematics department, Mkwawa University College of Education.

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1.3 Course Overview

MT200 is a course designed for second year students. The course comprises of function of several variables, which includes Jacobian matrix and determinant, applications in analysis, curves and regions, scalar and vector fields, orthogonal curvilinear coordinates, the definite Riemann integral in 3^n (n=2,3), vector integral calculus and integral theorems

1.4 Course Outcomes/Objectives

Generally at the end of the course you will be able to develop necessary concepts and techniques of differential integral calculus of several variables.

At the end of the course specifically, you will be able to:

· Define domain, limit, continuity, partial derivatives, and differentiability of functions of several variables.

· To determine domain, limits, partial derivatives, continuity and differentiability of functions of several variables.

· To calculate Jacobian matrix and determinant.

· To apply Lagrange multipliers to analyze extreme points.

· To represent space curves and regions in parametric form, and determine whether they are tangent /normal to the curve/surface.

· To find directional derivatives, gradient, del operator, divergence, laplacian and the curl of a vector field.

1.5 Pre-requisites

You will need the following prerequisite courses:

- MT120: Analysis1: Functions of single variable.
- MT127: Linear Algebra1.
- MT136: Ordinary differential equations I.

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1.6 Course Calendar/Schedule

WEEK	*LESSON*	*ACTIVITIES*	*FACILITATOR*
	Familiarization to the course	Registration Self Introduction	
1	Module 1 Functions of several variables L/Unit1: Domain, Limits and Continuity	Activity 1.1	J.I. Irunde

2	Module 1: Functions of several variables L/Unit2: Partial derivatives and differentiability. L/Unit3: Composite functions and Chain rule. Module 1: L/Unit3: Composite functions and Chain rule.	Activity: 1.2:	
	Module 2: Jacobian Matrix and determinant L/Unit4: Implicit functions and simultaneous implicit theorem. L/Unit5:Higher order derivative and Taylor's series	Activity: 2.1:	
3	Module3: Application in Analysis L/Unit6: Extrema, extrema with Constraints	Activity: 3.1	
	Module4: Curves and Region. L/Unit7&8: Space curve, parametric representation, piecewise smooth curves.	Activity: 4.1	
4	Module4: Curves and Region. L/Unit9: Orientation, length of the curve and regions. L\unit10: Simply-connected and multiply connected regions.	Activity: 4.2:	

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5	Module4: L/Unit12: Tangent and normal to curve, tangent plane and normal to surface. Module5: Scalar and vector fields. L/Unit13: Partial differentiation of vector function, directional derivative and gradient. L\Unit14: Del operator and its properties divModule5: L/Unit15: The curl, the physical interpretation and properties of divergence, laplacian and the curl..	Activity: 5.1:	
6	Module6: Orthogonal curvilinear Coordinates. L/Unit16: Transformation of coordinates, orthogonal curvilinear coordinates in space. L/Unit17: Unitary and Unit vectors in curvilinear systems- in particular rectangular.	Test1	
7	Module 6: Orthogonal curvilinear	Activity: 6.1	

	coordinates. L/Unit18: Cylindrical and spherical systems, Transformation of vectors between coordinate systems. L/Unit 12: Simple OHP maintenance		
8	8 Module 6: Orthogonal curvilinear coordinates L/Unit19: The gradient, divergence, curl and laplacian in orthogonal curvilinear coordinates L/Unit20: Arc length, element of arc length and volume element in general curvilinear coordinates (The metric form or fundamental quadratic form)	Activity: 6.2:	
9-10	Module7: The Definite Riemann integrals 3^n (n=2, 3). L/Unit21: Double and triple Riemann integral. L/Unit22: The iterated integrals, change of variables in integrals. L/Unit23: Arc length, volume and surface area.	Activity: 7.1:	
11-12	Module8. Vector integral calculus. L/Unit24: Line integral in the plane, integral with respect to arc length. L/Unit25: Line integral, integral of vectors and Green theorem. L/Unit26: Independence of path, line integral in space and surface in space.	Activity: 8.1&Test2	
12-15	Module8. Vector integral calculus	Activity 8.2&9.1&9.2	

	L\Unit27: Orientability, surface integral, and volume integral. Module9: Integral Theorems. L/Unit28: The Gauss and Stoke theorems. L/Unit29: Change of variables in multiple integrals and physical applications in dynamics, electromagnetism (etc):		
16-17	FINAL EXAMINATIONS THAT WILL BE CONDUCTED AT UDSM		

1.7 Course assessment and Grading Policy

WEEK NO	ASSESSMENT	MARKS (%)	REMARKS
6	Test1	20	Assessment based on content covered in weeks 1-5
12	Test2	20	Assessment based on content covered in week 6 -11
16-17	Examination	60	
	Total	100	

1.8 *References/resources.*

2 References

1. W. Kaplan: Advanced calculus Addis-Wesley. Co, New York 1991

2. Apostol T. Calculus Volume II, Volume II; John Wiley & Sons, 1969.

3.Bourne, D E& Kendall P C: *Vector Analysis and Cartesian Tensors*, Chapman and Hall, London, 1992.

4. Leithold, L: *The Calculus with Analytic Geometry,* Herper Collins Publisher,1990.

5. Spiegel M R. : *Advanced Calculus, Schaum Series*, Mc Graw Hill Book Co., New York, 1962.

2.1 Course notes/content

Modules:

- function of several variables, the Jacobian matrix and determinant, applications in analysis, curves and regions, scalar and vector fields, orthogonal curvilinear coordinates, the definite Riemann integral in 3^n (n=2,3), vector integral calculus and integral theorems.

MT200:CALCULUS OF SEVERAL VARIABLES

1 Function of several variables

Objectives for learning unit1

This learning unit covers domain, limits and continuity of the functions of several variables. The objectives of this learning unit are

(i) To define and identify the domain where the function is defined.

(ii) To sketch the domain.

(iii) To define and evaluate limits of the function.

(iv) To Use the limits to analyze the continuity of the function at a given point.

Functions of several variables refer to functions which involve more than one independent variables.This means that more than one independent variables are mapped to a single real number. If to each point (x, y) of xy-plane is assigned z, then z is said to be given as a function of the two real variables x and y. Consoder the following equations

$$z = x^2 - y^2. \tag{1}$$

$$z = x \sin xy. \tag{2}$$

$$u = xyz. \tag{3}$$

$$u = x^2 + y^2 + z^2 - t^2. \tag{4}$$

$$\tag{5}$$

Equations (1) and (2) are functions which involve two independent variables, equation (3) is a function which involves three independent variables and equation (4) is a function which involves four independent variables.

Functions of three or more variables differ slightly from functions of two variables. For this reason, the emphasis will be on the functions of two variables.

1.1 Domain and Range

A domain of a function is a set of independent variables on which a given function is defined (in many cases the domain is also refered to as a domain of definition).
A range of a function is a set of all resulting values of the dependent variables after substituting the values of the domain.

Example

Describe and sketch the domain of the following functions

(i) $f(x,y) = \dfrac{1}{x-2y}$

(ii) $g(x,y) = \sqrt{25-x^2-y^2}$

(iii) $arc\sin\left(x^2+y^2-2\right)$.

Solutions

(i) The function in item (i) can not be defined when $x = 2y$. Therefore the function is defined when $x \neq 2y$, hence the Domain=$\{(x,y)\colon\ x \neq 2y\}$

(ii) For the function to be defined, it requires that

$$25-x^2-y^2 \geq 0,$$
$$x^2+y^2 \leq 25.$$

Domain=$\{(x,y)\colon\ x^2+y^2 \leq 25\}$

1.2 Limits and continuity

1.2.1 Limits

Let $z = f(x, y)$ be given in a domain D, and let (x_1, y_1) be a point in D, then the equation

$$\lim_{x \to x_1, y \to y_1} f(x, y) = L \tag{6}$$

means that, given $\epsilon > 0$. a $\delta > 0$ can be found such that for every (x, y) in D and with neighbourhood (x, y) of radius δ, one has

$$0 < \sqrt{(x - x_1)^2 + (y - y_1)^2} < \delta \longrightarrow |f(x, y) - L| < \epsilon \tag{7}$$

Thus if the variable point (x, y) is sufficiently close to its limiting position (x_1, y_1), the value of the function is as close as desired to limiting value L. Consider the following example where we will use $\epsilon - \delta$ to prove the limit.

Example

Use the definition of limits to prove that

$$\lim_{(x,y) \to (1,3)} 2x + 3y = 11 \tag{8}$$

Proof.

We need to show that $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$0 < \sqrt{(x - 1)^2 + (y - 3)^2} < \delta \longrightarrow |2x + 3x - 11| < \epsilon. \tag{9}$$

Using triangular inequality, we have

$$\begin{aligned} &|2x + 3x - 11| = |2x - 2 + 3x - 9|, \\ &= |2(x - 1) + 3(y - 3)|, \\ &\leq 2|x - 1| + 3|y - 3|. \end{aligned}$$

Since $|x - 1| < \sqrt{(x - 1)^2 + (y - 3)^2}$ and $y - 3 < \sqrt{(x - 1)^2 + (y - 3)^2}$, then it follows that $|x - 1| < \delta$ and $|y - 3| < \delta$.

$$0 < \sqrt{(x - 1)^2 + (y - 3)^2} < \delta \longrightarrow |2x + 3x - 11| < 2\delta + 3\delta = 5\delta.$$

From (9), the suitable choice for δ is $5\delta = \epsilon \longrightarrow \delta = \dfrac{\epsilon}{5}$. This completes the proof.

1.2.2 Evaluation of Limits

Computation of limits for functions of several variables involve direct substitution of values of variables into the function. However, the direct substitution is not always possible for rational functions, as this may result into indeterminate (undefined) form. If this form is obtained, we have to employ other techniques that will enable us to determine the required limits if they exist. The quickest technique is to factorize both numerator and denominator and cancel out the common factors. If this is also impossible we have to use the following techniques;

(i) Different paths method.

(ii) General path method.

1.2.3 Different paths method

In this method one can consider two different paths, for example $y = x$, $y = 0$. Limit of a function is said to exist if the limiting values for both paths taken are the same. Otherwise the limit does not exist.

Example

Evaluate the limit of the following function if it exists;

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} \tag{10}$$

Solution

When x and y are directly substituted in the equation we have

$$\frac{(0)(0)}{0^2 + 0^2} = \frac{0}{0}$$

which gives indeterminate form, therefore we use different paths method;
Let's consider two paths $y = 0$ and $y = x$,

$y = 0 \to \lim\limits_{x\to 0} 0 = 0.$

Consider $y = x$,

$y = x \to \lim\limits_{x\to 0} \frac{1}{2} = \frac{1}{2}.$

Since $0 \neq \frac{1}{2}$, the limit does not exist.

1.2.4 General path method

In this method, one path $y = mx$ is considered. The limit of a given function will exist if for any choice of m (where m is any natural number), the limiting values of the function are the same.

1.2.5 Properties of Limits

The following theorem will help us to understand the properties of limits' operations.

Theorem 1 *Let $u = f(x, y)$ and $v = g(x, y)$ both be defined in the domain D of the xy-plane. Let*
$\lim_{(x,y)\to(x_1,y_1)} f(x, y) = u_1$, $\lim_{(x,y)\to(x_1,y_1)} g(x, y) = v_1$. *Then*

(i) $\lim_{(x,y)\to(x_1,y_1)} [f(x, y) + g(x, y)] = u_1 + v_1$.

(ii) $\lim_{(x,y)\to(x_1,y_1)} [f(x, y).g(x, y)] = u_1.v_1$.

(iii) $\lim_{(x,y)\to(x_1,y_1)} \left[\frac{f(x, y)}{g(x, y)}\right] = \frac{u_1}{v_1}$.

1.3 Continuity of function

The function of two variables (x, y) is said to be continuous at a point (x_0, y_0) if and only if the following three conditions are satisfied;

(i) $f(x_0, y_0)$ is defined.

(ii) $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists.

(iii) $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$

Example1
Discuss the continuity of $f(x, y)$ at $(0, 0)$ if

$$f(x, y) = \begin{cases} \frac{3x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 \text{ if } (x, y) = (0, 0). \end{cases}$$

Solution
Using the three conditions for continuity, Condition (i) holds $f(0, 0) = 0$ which shows that the function is defined at a point $(0, 0)$.

To determine whether the limit of the function exists, we use general path method

Let $y = mx$,

$$\lim_{x\to 0} \frac{3mx^3}{x^2(1+m^2)} = \lim_{x\to 0} \frac{3mx}{(1+m^2)} = 0,$$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{3mx^3}{x^2(1+m^2)} = f(0,0).$$

Therefore the limit exists and function is continuous at a point $(0, 0)$.

MT200:CALCULUS OF SEVERAL VARIABLES

1 Function of several variables

Objectives for learning unit2
This learning unit covers Partial derivatives and differentiability of the functions of several variables.
The objectives of this learning unit are

(i) To define and compute the partial derivatives of the functions of several variables.

(ii) To define differentiability and determine whether the given function is differentiable.

1.1 Partial derivatives

Let $z = f(x, y)$ be defined in a domain D and let (x_1, y_1) be a fixed point. The function $f(x, y_1)$ then depends on x alone and is defined in an interval about x_1. Hence its derivative with respect to x at $x = x_1$ may exist.If it exists it is called the partial derivative of $f(x, y)$ with respect to x at (x_1, y_1) and is denoted by; $\dfrac{\partial f}{\partial x}(x_1, y_1)$ or $\dfrac{\partial z}{\partial x}(x_1, y_1)$. Thus by definition of derivative, one has

$$\frac{\partial f}{\partial x}(x_1, y_1) = \frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y_1) - f(x_1, y_1)}{\Delta x}, \tag{1}$$

The partial derivative $\dfrac{\partial f}{\partial y}(x_1, y_1)$ is defined similarly, we now holds x constant and differentiate $f(x_1, y)$ with respect with y. Using the definition we have;

$$\frac{\partial f}{\partial y}(x_1, y_1) = \frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x_1, y_1 + \Delta y) - f(x_1, y_1)}{\Delta y}. \tag{2}$$

The notations f_x, f_y or z_x, z_y are commonly used for partial derivatives with respect to x and y respectively. The partial derivatives $\frac{\partial f}{\partial x}(x_1, y_1)$ and $\frac{\partial f}{\partial y}(x_1, y_1)$ are interpreted as slopes of the tangent to the curve at which both planes $x = x_1$ and $y = y_1$ cut the surface(since $z = f(x, y)$ represents a surface).
If the point (x_1, y_1) is now varied, we obtains a new function of two variables, the function $f_x(x, y)$, similarly the functions $f_y(x, y)$. This definition can be extended to functions of three or more variables.
If $x^2 + y^2 - z^2 = 1$,then

Solution

$2x - 2z\frac{\partial z}{\partial x} = 0,\ 2y - 2z\frac{\partial z}{\partial y} = 0,$
$\frac{\partial z}{\partial x} = \frac{x}{z},\ \frac{\partial z}{\partial y} = \frac{y}{z}(\text{z} \neq 0).$

Activity

(i) If $w = xuv + u - 2v$, obtain w_x, w_u, w_v.

(ii) Given $f(x, y) = x^2 + 3y$, find f_x and f_y in terms of limits.

1.2 Differentiability

In forming partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$, the changes Δx and Δy in x and y were considered separately. Now we consider the effect of changing x and y together at the same time. Let (x, y) be a fixed point in D and let $(x + \Delta x, y + \Delta y)$ be a second point in D, then the function $z = f(x, y)$ changes by an amount Δz in going from (x, y) to $(x + \Delta x, y + \Delta y)$, therefore

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) \tag{3}$$

Δz is defined as a function of Δx and Δy with property that $\Delta z = 0$ when $\Delta x = 0$ and $\Delta y = 0$. For example, if

$$z = x^2 + xy + xy^2, \tag{4}$$

then

$$\begin{aligned}
&\Delta z = (x + \Delta x)^2 + (x + \Delta x)((y + \Delta y) + (x + \Delta x)((y + \Delta y)^2 - x^2 - xy - xy^2,\\
&= 2x\Delta x + (\Delta x)^2 + x\Delta y + y\Delta x + \Delta x\Delta y + 2xy\Delta y + x(\Delta y)^2 + y^2\Delta x + x(\Delta y)^2\\
&+2y\Delta x\Delta y + \Delta x(\Delta y)^2,\\
&= \Delta x(2x + y + y^2) + \Delta y(x + 2xy) + (\Delta x)^2 + \Delta x\Delta y(1 + 2y) + x(\Delta y)^2 + \Delta x(\Delta y)^2.
\end{aligned} \tag{5}$$

Here Δz can be expressed in the form

$$\Delta z = a\Delta x + b\Delta y + c(\Delta x)^2 + e(\Delta y)^2 + f\Delta x(\Delta y)^2, \quad \text{where,} \tag{6}$$

This is a linear function of Δx and Δy plus terms of higher degree. In general, the function $z = f(x, y)$ is said to be differentiable at the point (x, y) if

$$\Delta z = a\Delta x + b\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y, \quad \text{where,} \tag{7}$$

a and b are independent of Δx and Δy and ϵ_1 and ϵ_2 are functions Δx and Δy such that $\epsilon_1 \to 0 \quad \text{and} \quad \epsilon \to 0 \quad \text{as} \quad (\Delta x, \Delta y) \to (0, 0)$.
The linear function of Δx and Δy

$$a\Delta x + b\Delta y$$

is termed as the total differential of z at the point (x, y) and is denoted by

$$dz = a\Delta x + b\Delta y \tag{8}$$

If Δx and Δy are sufficiently small, dz gives a close approximation to Δz

$$\Delta z = \Delta x(a + \epsilon_1) + \Delta y(y + \epsilon_2)$$

where a and b are constants. In the above example Δz has a total differential at each point (x, y), with

$$a = 2x + y + y^2, \quad b = x + 2xy, \quad \text{and} \quad \epsilon_1 = \Delta x + \Delta y(1 + 2y), \quad \epsilon_2 = x\Delta y + \Delta x\Delta y.$$

The results above can be summarized in the following theorem

Theorem 1 *If $z = f(x, y)$ has a total differential $dz = a\Delta x + b\Delta y$ at a point (x, y) then f is continuous at (x, y) and $a = \dfrac{\partial z}{\partial x}$, $b = \dfrac{\partial z}{\partial y}$.*

Proof

Set $\Delta y = 0$.Then by

$$\Delta z = a\Delta x + b\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \quad \text{and}$$

$\lim_{\Delta x \to 0} \epsilon_1 = 0 \quad, \lim_{\Delta y \to 0} \epsilon_2 = 0.$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(a + \epsilon_1)}{\Delta x} = \lim_{\Delta x \to 0} (a + \epsilon_1) = a. \tag{9}$$

Similarly by setting $\Delta x = 0$ we can prove that $\dfrac{\partial z}{\partial y} = b$. This completes the proof.
If $z = f(x, y)$ has continuous first partial derivatives in D, then it is possible one to prove that z has a differential

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y \tag{10}$$

at every point (x, y).
Proof
Let (x, y) be a fixed point in D, if x alone changes, one obtains a change Δz in z

$$\Delta z = f(x + \Delta x, y) - f(x, y),$$

this difference can be evaluated by the law of the mean for functions of one variable,for y held fixed, z is a function of x having continuous derivative $f_x(x, y)$. Thus

$$f(x + \Delta x, y) - f(x, y) = f_x(x_1, y)\Delta x, \quad \text{where} \quad x \leq x_1 \leq x + \Delta x. \tag{11}$$

Since $f_x(x, y)$ is continuous, the difference

$$\epsilon_1 = f_x(x_1, y) - f_x(x, y), \tag{12}$$

approaches to 0 as $\Delta x \to 0$, thus

$$f(x + \Delta x, y) - f(x, y) = f_x(x, y)\Delta x + \epsilon_1 \Delta x. \tag{13}$$

Now if both x and y change , we obtain a change Δz in z

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) \tag{14}$$

This can be written as a sum of terms representing the effect of change in x alone and subsiquent change in y alone.

$$\Delta z = [f(x + \Delta x, y) - f(x, y)] + [\Delta z = f(x, y + \Delta y) - f(x, y)] \tag{15}$$

$$f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) = f_y(x,y)\Delta y, \quad \text{where,} \quad y \leq y_1 \leq y+\Delta y, \tag{16}$$

It follows that from continuity of $f_y(x,y)$ that the difference

$$\epsilon_2 = f_y(x+\Delta x, y_1) - f_y(x,y) \tag{17}$$

approaches to 0 as $\Delta y \to 0$. Now Δz gives

$$\Delta z = f_x(x,y)\Delta x + f_y(x,y)y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \tag{18}$$

where $\lim\limits_{(\Delta x, \Delta y)\to(0,0)} \epsilon_1 = 0, \quad \lim\limits_{(\Delta x, \Delta y)\to(0,0)} \epsilon_2 = 0$. Thus z has a differential dz as stated in(10) and hence it is proved. Δx and Δy can be replaced by dx and dy thus we have

$$dz = f_x(x,y)dx + f_y(x,y)dy. \tag{19}$$

This is a common way of writing differentials. This notion can be extended to functions of three or more variables.For instance, given

$$\begin{gathered} w = f(x,y,u,v), \text{We} \quad \text{have} \quad \text{a} \quad \text{differential,} \\ dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial u}du + \frac{\partial w}{\partial v}dv. \end{gathered} \tag{20}$$

Given

$$z = x^2 - y^2, \quad \text{and} \quad w = \frac{xy}{z}$$

Find dz and dw.

Solution
$dz = 2xdx - 2ydy \quad \text{and} \quad dw = \frac{y}{z}dx + \frac{x}{z} - \frac{xy}{z^2}dz.$
The partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are in general functions of x and y, they may be differentiated.

$$\frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial x^2}, \qquad \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial y^2}.$$

$$\frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y \partial x}, \qquad \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial x \partial y}.$$

$\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are called mixed derivatives.

Theorem 2 *Given a function of several variables $f(x,y)$. If both mixed partial derivatives are continuous then they are equal. ie the order of differentiation is not important.*

Example6

Find mixed partial partial derivatives of the function

$$f(x, y) = x^5 + 4x^3y - 5xy^2.$$

Solution

$$\frac{\partial f}{\partial x} = 5x^{5-1} + 3(4)x^{3-1}y - (1)5x^{1-1}y^2 = 5x^4 + 12x^2y - 5y^2,$$

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} = 12x^2 - 10y,$$

$$\frac{\partial f}{\partial y} = (1)4x^3y^{1-1} - 2(5)xy^{2-1} = 4x^3 - 10xy,$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = 12x^2 - 10y.$$

Activity

Verify that $f_{xy} = f_{yx}$ for the following function;

$$f(x, y) = \frac{x}{x + y}.$$

MT200:CALCULUS OF SEVERAL VARIABLES

1 Function of several variables

Objectives for learning unit3
This learning unit covers composite and chain rule for the functions of several variables. The objective of this learning unit is

(i) To apply chain rule to compute derivatives of a composite functions of severals variables.

1.1 Composite functions and chain rule

1.1.1 Composite functions

Let $u = f(x, y)$, $x = g(r, s)$ and $y = h(r, s)$, u can be expressed in terms of r and s

$$u = f(x, y) = f(g(r, s), h(r, s)). \tag{1}$$

The corresponding chain rule need the concept of differential.
From

$$u = f(x, y), x = g(r, s), y = h(r, s)$$

Then

$$du = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy,$$

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial s}ds, \tag{2}$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial s}ds.$$

substitute for dx and dy into du.

$$du = \frac{\partial f}{\partial x}(\frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial s}ds) + \frac{\partial f}{\partial y}(\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial s}ds),$$

$$du = (\frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r})dr + (\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s})ds. \quad (3)$$

Since u is finally a function of r and s, we have

$$du = \frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial s}ds. \quad (4)$$

Compare equations (3) and (4) we obtain

$$\frac{\partial u}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r},$$

$$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}. \quad (5)$$

Example 8

$$f(x, y) = x^2 - 4y^2, \quad x = r\cos t, \quad y = r\sin t. \quad (6)$$

Find the partial derivatives of f with respect to r and t.
However, the chain rule can be extended for more than two variables. For the case of three variables, such as

$$u = f(x, y, z), \quad x = g(r, s), \quad y = h(r, y), \quad z = k(r, s), \quad \text{then,}$$

$$\frac{\partial u}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}, \quad (7)$$

$$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}.$$

Example 9
Find the partial derivatives of f with respect to r and s, given

$$u = f(x, y, z) = \sqrt{2xy^2 + 6z^2}, \quad x = r\cos 2s, \quad y = -r\sin s, \quad z = re^s.$$

1.1.2 The Chain rule

When we are dealing with two sets of functions

$$\begin{aligned} y_1 &= f_1(u_1.....u_p), \\ &. \\ &. \\ &. \\ y_m &= f_m(u_1.....u_p) \end{aligned} \quad (8)$$

and

$$
\begin{aligned}
&u_1 = g_1(x_1.....x_n),\\
&.\\
&.\\
&.\\
&u_p = g_p(x_1.....x_n)
\end{aligned} \tag{9}
$$

When functions $u_1.....u_p$ are substituted in the functions $y_1.....y_m$, one obtains composite functions

$$
\begin{aligned}
&y_1 = f_1(g_1(x_1.....x_n).....g_p(x_1.....x_n)) = F_1(x_1.....x_n),\\
&.\\
&.\\
&.\\
&y_m = f_m(g_1(x_1.....x_n).....g_p(x_1.....x_n)) = F_m(x_1.....x_n).
\end{aligned} \tag{10}
$$

When we use Chain rule one can obtain the partial derivatives as we did in composite functions

$$\frac{y_i}{\partial x_j} = \frac{\partial y_i}{\partial u_1}\frac{\partial u_1}{\partial x_j} + ... + \frac{\partial y_i}{\partial u_p}\frac{\partial u_p}{\partial x_j} \quad (i = 1....m, \quad j = 1....n). \tag{11}$$

Thus the formula can be expressed in matrix in which the partial derivatives $\dfrac{\partial y_i}{\partial x_j}$ are the entries in the mxn matrix

$$\frac{\partial \mathbf{y_i}}{\partial \mathbf{x_j}} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & . & . & . & . & \frac{\partial y_1}{\partial x_n} \\ . & . & . & . & . & . & \\ . & . & . & . & . & . & \\ . & . & . & . & . & . & \\ . & . & . & . & . & . & \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & . & . & . & . & \frac{\partial y_m}{\partial x_n} \end{pmatrix} \tag{12}$$

This is the Jacobian matrix of the mapping (11). $\dfrac{\partial y_i}{\partial x_j}$ involves two other Jacobian matrices

$$\frac{\partial \mathbf{y_i}}{\partial \mathbf{u_j}} = \begin{pmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} & . & . & . & . & \frac{\partial y_1}{\partial u_p} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \frac{\partial y_m}{\partial u_1} & \frac{\partial y_m}{\partial u_2} & . & . & . & . & \frac{\partial y_m}{\partial u_p} \end{pmatrix} \tag{13}$$

and

$$\frac{\partial \mathbf{u_i}}{\partial \mathbf{x_j}} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & . & . & . & \frac{\partial u_1}{\partial x_n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \frac{\partial u_p}{\partial x_1} & \frac{\partial u_p}{\partial x_2} & . & . & . & \frac{\partial u_p}{\partial x_n} \end{pmatrix} \tag{14}$$

The Chain rule states that the product of $\dfrac{\partial y_i}{\partial u_j}$ and $\dfrac{\partial u_i}{\partial x_j}$ is equal to $\dfrac{\partial y_i}{\partial x_j}$.

$$\frac{\partial y_i}{\partial x_j} = \frac{\partial y_i}{\partial u_j}\frac{\partial u_i}{\partial x_j}. \tag{15}$$

This equation is called the general Chain rule.

Example

$$\begin{aligned} &y_1 = u_1u_2 - u_1u_3, \quad y_2 = u_1u_3 + u_2^2, \\ &u_1 = x_1\cos x_2 + (x_1 - x_2)^2, \quad u_2 = x_1\sin x_2 + x_1x_2, \quad u_3 = x_1^2 - x_1x_2 + x_2^2. \end{aligned} \tag{16}$$

Obtain $\dfrac{\partial y_i}{\partial x_j}$ matrix.

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DEPARTMENT OF MATHEMATICS

MT200: CALCULUS OF SEVERAL VARIABLES

TUTORIAL SHEET1

1. For the following functions describe and sketch the domain
   (a) $z_1 = \sqrt{9 - x^2} + \sqrt{y^2 - 4}$.
   (b) $z_2 = \ln\big((36 - x^2 - y^2)(x^2 + y^2 - 9)\big)$.

2. State the range for the following functions
   (a) $f(x, y) = \sqrt{16 - x^2 - y^2}$.
   (b) $g(x, y) = \sqrt{25 - x^2 - y^2}$.

3. Evaluate the limits of the folowing functions if they exist
   (a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2 + y^2}$.
   (b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4 - y^4}{x^2 + y^2}$.

4. Discuss the continuity of $f(x, y)$ at $(0, 0)$ if

$$f(x,y)=\begin{cases} \dfrac{3x^2y}{x^2+y^2} \text{ if } (x,y) \neq (0,0) \\ 0 \text{ if } \quad \text{(x,y)=(0,0).} \end{cases}$$

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DEPARTMENT OF MATHEMATICS

MT200: CALCULUS OF SEVERAL VARIABLES

TUTORIAL SHEET2

1. (a) $z = y \sin xy$,

   (b) $z = arc \sin(x + 2y)$

   Determine $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

2. If $z = \dfrac{xy}{1 - x - y}$ use the definition of partial derivative to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

3. Evaluate $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial y}$ if $f(x, y) = \sqrt{e^{x+2y} - y^2}$.

4. Given $f(x, y) = x^2 + 3y$, find f_x and f_y in terms of limits.